Least-Weighted-Square Method for Analysis and Synthesis of Transmission Lines

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Abstract—Analysis and synthesis of transmission lines of arbitrary geometry are not easy to realize with currently available methods. The object of this paper is to show that by introducing a new principle to be called the "least-weighted-square invariance deformation," it is possible to solve transmission line problems to desired orders of accuracy. A procedure based on this principle is given for deforming a given transmission line geometry keeping the characteristic impedance an invariant, or for synthesizing a transmission line cross-sectional geometry corresponding to the given constant parameters of the structure. The method is applied to the analysis of transmission lines having a regular polygonal outer conductor and a circular inner conductor. An application to a representative situation in synthesis is also described.

I. Introduction

THOUGH TRANSMISSION lines have been analyzed exactly for a few simple geometries, there appears to be no general procedure for obtaining explicit analytical expressions for parameters like the characteristic impedance when the geometry is arbitrary and involves many conductors. The synthesis problem, that of designing a suitable cross-sectional geometry for the simultaneous specification of the values of certain electrical and mechanical parameters, has not been investigated so far. The object of this paper is to describe a method based on a new principle to be called "least-weighted-square invariance deformation" and to apply it to problems in both analysis and synthesis. Though the parameter considered throughout the paper is the characteristic impedance, the principle itself is of wider applicability.

The subject matter of the paper is treated in the following order. The least-weighted-square deformation principle is first described, showing how a structure can be deformed from one shape to another without changing the characteristic impedance per unit length or any other unique parameter of the structure. This method is applied to the analysis of a transmission line with a circular inner conductor and a regular polygonal outer conductor, and expressions for the characteristic impedance and the TEM wave are derived. Section IV gives a description of the synthesis problem from the field theoretic viewpoint as encountered in transmission line theory. Methods are given for designing transmission line geometry for the specified electrical and mechanical requirements. In the last section, an extension of the methods of the previous section to transmission lines with more than two conductors is discussed. In the Appendix, a justification for the theorem utilized in the main body of the paper is given, together with the description of a method for evaluating certain universal constants associated with transmission lines.

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II. Least-Weighted-Square Deformation Principle

It may be required, sometimes, to analyze an electromagnetic structure whose boundary configuration is very nearly that of one which has been analyzed previously. Situations may also arise wherein a structure to be analyzed, on being conformally transformed by known analytic functions, may come close to a known analyzed structure. In such cases, it is possible to find the dimensions of the analyzed structure that has the same characteristic impedance as the given structure, which has not been analyzed before, by using the methods based on the principle of least-weighted-square deformation explained below.

Description in the present section is confined to the case of determining the characteristic impedance of two, long, open plates and of TEM structures with a circular inner conductor and an outer conductor of arbitrary shape capable of being described by a power series. It should be noted that when both conductors are of arbitrary geometry, the following method is still useful, though it may be required to deform each conductor one after the other.

In Fig. 1(a) or (b), curve A_0 is the boundary which remains unchanged during the transformation; curve A_1 is the boundary to be deformed; and curve A_2 is the deformed form of curve A_1 such that the characteristic impedance between A_1 and A_0 is the same as that between A_2 and A_0. If A_{0j} and A_{1j} are known constants, and A_{2j} are undetermined coefficients, then the three functions describing A_1, A_2, A_3 are, in Cartesian coordinates (x, y),

$$y_i = \sum_{j=1}^{t} A_{ij}x^{j-1}; \qquad (x_{A_i1} \leq x \leq x_{A_i2}; i = 0, 1, 2), \quad (1)$$

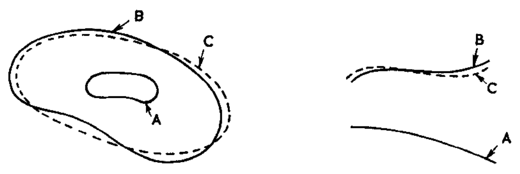
or in polar coordinates (r, θ),

$$r_i = \sum_{j=1}^{t} A_{ij}\theta^{j-1}; \qquad (\theta_{A_i1} \leq \theta \leq \theta_{A_iz}; i = 0, 1, 2). \quad (2)$$

By representing A_{2j} as explicit functions of not more than t parametric variables, retaining at least a single degree of freedom, it is possible to choose the particular form of the perturbed boundary though not the actual dimensions. Thus, for a maximum of t degrees of freedom, the parametric form of the coefficients will be

$$A_{2j} = f_j(\xi_1, \xi_2, , \cdots, \xi_t); \qquad j = 1, 2, \cdots, t. \quad (3)$$

The deformation of curve A_2 to curve A_3, in such a way as to keep the characteristic impedance an invariant under



(a) (b)

Fig. 1. Transformation of boundaries in transmission lines with two conductors.

such a deformation, is described in two steps. A theorem for the existence of a weighting function governing the least-weighted-square principle is stated first. This is followed by the derivation of explicit expressions for the undetermined coefficients A_{2j}, together with a method for evaluating $\xi_1, \xi_2, \cdots, \xi_t$.

The theorem requires the definition of the following five terms.

Definition 1

The *least Newtonian path length* is the shortest distance between two given points, one on each of two boundary curves, such that the entire path is within the annular region.

Definition 2

Two boundaries are *mutually compatible* if the end points of the least Newtonian path trace both the boundaries continuously and entirely.

Definition 3

A *structure-parameter set* of an n-dimensional structure is a set of n independent parameters which determines the size, shape, and dimensions of the structure uniquely.

Definition 4

A *superficial magnification* is defined to be possible if the structure parameters remain invariant under any proportionate change in the dimensions of the entire structure.

Definition 5

Weighting function of the least Newtonian path length is a function deciding the relative deformations of various sections of a curve representing a boundary that is being deformed by the least-square criteria [1].

Theorem

A deformation of one of the boundaries of a transmission line, keeping the other boundaries unperturbed, leaves the structure parameters that maintain invariancy under a superficial magnification, invariant under the deformation if the sum of weighted squares of ordinate (or radius vector) deviations, represented by

$$W = \int_{x_1}^{x_2} \left[\left(\sum_{j=1}^{t} A_{2j} x^{j-1} - \sum_{j=1}^{t} A_{3j} x^{j-1} \right) \frac{1}{F[d(x)]} \right]^2 dx \tag{4}$$

(in Cartesian coordinates)

or

$$W = \int_{0}^{2\pi} \left[\left(\sum_{j=1}^{t} A_{2j} \theta^{j-1} - \sum_{j=1}^{t} A_{3j} \theta^{j-1} \right) \frac{1}{F[d(\theta)]} \right]^2 d\theta, \tag{5}$$

(in polar coordinates)

is a minimum, where $F[d(x)]$ or $F[d(\theta)]$, which is a weighting function of the least Newtonian path length $d(x)$ or $d(\theta)$ corresponding to any x or θ, respectively, exists for mutually compatible, power series describable boundaries.

A justification for this theorem is given in the Appendix. It is shown there that

1) there exists a set of universal dimensionless constants defining the weightages $F[d(x)]$ or $F[d(\theta)]$;
2) for structures governed by Laplace's equation or for those in the TEM mode, a weighting function is given by

$$F[d(x)] = [1/d(x)]^s \quad \text{or} \quad F[d(\theta)] = [1/d(\theta)]^s, \tag{6}$$

where s is a universal constant (i.e., independent of the shape, size, or dimensions of the structure) and given by

$$s = 1.202 \text{ (Cartesian)}$$
$$= 1.251 \text{ (polar)}.$$

It was observed in the above theorem that (4) or (5) should be a minimum. This is true if, on substituting (3) into (4) or (5),

$$\partial W / \partial \xi_k = 0; \quad k = 1, 2, \cdots, t.$$

Since C_j are explicit functions of ξ_k, this condition is equivalent to

$$\partial W / \partial C_j = 0; \quad j = 1, 2, \cdots, t \tag{7}$$

subjected to the conditions of (3). Consequently, ξ_k can be evaluated. By a well-known procedure attributed to Gauss (see Sokolnikoff and Redheffer [1]) the modified (weighted) normal equations can be derived from (6), (7), and (4) or (5). The normal equations in Cartesian coordinates, using matrix notations, are

$$\|C_j\|_j^T = \left\| \int_{x_1}^{x_2} \frac{x^{i+j-2}\, dx}{[d(x)]^{2s}} \right\|_{j,i=1,2,\cdots,t}^{-1} \left\| \int_{x_1}^{x_2} \frac{y_1 x^{i-1}\, dx}{[d(x)]^{2s}} \right\|_i^T. \tag{8}$$

Equation (8) can be written in the "limit of a sum" form as

$$\|C_j\|_j^T = \left\| \lim_{m \to \infty} \sum_{l=1}^{m} \left\{ \frac{x_l^{i+j-2}}{[d(x_l)]^{2s}} \right\} \right\|_{j,i=1,2,\cdots,t}^{-1} \left\| \lim_{m \to \infty} \sum_{l=1}^{m} \left\{ \frac{y_{1l} x_l^{i-1}}{[d(x_l)]^{2s}} \right\} \right\|_i^T \tag{9}$$

where y_{1l} is equal to the value of y_1 at x_l, and, similarly, in polar coordinates as

$$\|C_j\|_j^T = \left\| \lim_{m \to \infty} \sum_{l=1}^{m} \frac{\theta_l^{i+j-2}}{[d(\theta_l)]^{2s}} \right\|_{j,i=1,2,\cdots,t}^{-1} \left\| \lim_{m \to \infty} \sum_{l=1}^{m} \frac{r_{1l} \theta_l^{i-1}}{[d(\theta_l)]^{2s}} \right\|_i^T \tag{10}$$

where r_{1l} is equal to the value of r_1 at θ_l.

In the next section, the preceding method is applied to the analysis of a transmission line with circular inner conductor and a regular polygonal outer conductor, and the results are verified by comparing them with those obtained by other methods.

III. Application to the Analysis of Polygon-Circular Transmission Lines

In this section, it is intended to show that the least-weighted-square deformation can be applied in conjunction with known conformal transformations to arrive at the characteristic impedance or the TEM wave equation explicitly in terms of the physical dimensions of the transmission line. As an example, a transmission line with a circular inner conductor, a regular polygonal outer conductor, and a perfect dielectric annulus is taken. For this family of transmission lines an explicit relation for the characteristic impedance is available only for the square-circle configuration, whereas the TEM wave equation has not been derived for any member of this family. It should be noted in the following derivations that among the regular $n$gons considered, $n=2$ does not correspond to parallel planes but to an ellipse having finite major diameter. This structure has been analyzed by the author in previous communications [2], [3]. Therefore, in what follows, only those structures with $n=3$ to ∞ are considered. The first part is confined to the derivation of the characteristic impedance of this family, and in the second part the TEM equations are derived.

A. Characteristic Impedance

A conformal transformation $W=Z^n$ maps the $n$gon of side $2a$ and an inner conductor of radius r_i onto n identical Riemann sheets [4], each sheet having a circular inner boundary of radius r_i^n and an outer boundary described in polar form (R, Θ) by

$$R(a, n, \Theta) = a^n \cot^n (\pi/n) \sec^n \left(\frac{\pi - \Theta}{n}\right); \quad \Theta = 0 \text{ to } 2\pi \text{ radians}. \tag{11}$$

Each of the n sheets can be deformed, using the least-weighted-square principle, into an elliptic outer conductor and a noncoaxial circular inner conductor. Let this ellipse have a semimajor axis of a_0, semiminor axis b_0, and let the distance between its geometric center and the center of the inner circle of radius r_0 be e_0. If we let

$$\xi_1 = b_0/a_0;\ \xi_2 = a_0;\ \xi_3 = e_0/a_0;\ C_1 = \xi_1^2\xi_2(1-\xi_3^2); \quad C_2 = -2\xi_1^2\xi_2\xi_3;\ C_3 = -\xi_1^2;\ x = R\sin\Theta; \quad w = (R\cos\Theta)^2; \tag{12}$$

then (11) can be modified into

$$w = C_1 + C_2 x + C_3 x^2. \tag{13}$$

For (13), the normal equation corresponding to (9) will be

$$\|C_j\|_j^T = \left\| \sum_{l=1}^{m} \frac{x_l^{i+j-2}}{d_l^{2.404}} \right\|_{j,i=1,2,3}^{-1} \left\| \sum_{l=1}^{m} \frac{w_l x_l^{i-1}}{d_l^{2.404}} \right\|_i^T. \tag{14}$$

In (14), m is equal to the number of points chosen to approximate the locus of (11). If characteristic impedance is desired for $0<r_i/a<0.6$, a further approximation can be made, i.e., the least Newtonian path length d_l is the distance between the center of the inner circle and the lth point on the locus of (11). With these data it is possible to evaluate C_1, C_2, and C_3 from (14). By substituting these values in (12), the numerical values of a_0, b_0, and e_0 can be realized. Table I gives the values of ξ_1, ξ_2, and ξ_3 for $n=3$, 4, 5, and 6. More accurate values can be obtained by taking the least Newtonian path length d_l as the difference between the above value of d_l and r. In the above, the problem of determining the characteristic impedance of a regular polygon-circular line is reduced to the problem of finding that for a noncoaxial elliptic-circular line. The latter was the subject matter of an earlier communication by the author [2] and, therefore, the derivation obtained therefrom can be directly utilized for expressing the characteristic impedance Z_0 of a polygon-circular coaxial line. Therefore, if z is the ratio of the diameter of the inscribed circle of the regular $n$gon to the diameter of the inner conductor, then

$$Z_0 = 60 \frac{\cosh^{-1}\left(\dfrac{1+m^2(1-\xi_3^2)}{2m}\right)\cosh^{-1}\left(\dfrac{\xi_1^2 m^4+1}{m^2(1+\xi_1^2)}\right)}{2n(1nm)\left(1+\dfrac{1}{2zn^2}\right)} \text{ ohms,} \tag{15}$$

where

$$m = z^n/(1-\xi_3). \tag{16}$$

A more useful form for Z_0 will be the asymptotic formula

$$Z_0 = 60 \ln (A_n z) \text{ ohms}; \qquad z \gg 1 \tag{17}$$

where A_n should be evaluated for each n. From (15) and (17), we obtain an expression for A_n, viz.,

$$A_n = 1/Z \cdot \exp\left[\frac{\cosh^{-1}\left(\dfrac{1+m^2(1-\xi_3^2)}{2m}\right)\cosh^{-1}\left(\dfrac{\xi_1^2 m^4+1}{m^2(1+\xi_1^2)}\right)}{2n(1nm)\left(1+\dfrac{1}{2zn^2}\right)}\right]. \tag{18}$$

In (18), it can be observed that as n tends to infinity, A_n tends to unity, which is as it should be since this case represents concentric circles. It should be noted that for each n, A_n need not be a constant for all values of z. However, it was observed that except for $n=3$, A_n was a constant to within the third decimal place for $n=4$, 5, 6 in the range $2<z<4$. Table II gives the values of A_n for different combinations of z and n.

The formula for Z_0 of (15) was sought to be verified by comparing it with the results obtained by other methods. For $n=4$, a well-known formula [5], which is accurate to within 1 percent at 50 ohms, is available. The values of Z_0 derived from (15) are compared with that derived from the preceding known formula. Fig. 2 gives the relative error between the two formulas for $n=4$. In Figs. 3, 4, 5, and 6 the characteristic impedance per unit length as calculated from (15) is plotted for $n=3$, 4, 5, and 6, respectively, along with the characteristic impedance for the inscribed and escribed circles of the regular polygon.

TABLE I

Constants for the Polygon-Circle Coxial Line

n	ξ_1	ξ_3	ρ
3	0.628	0.696	0.892
4	0.769	0.506	0.935
5	0.861	0.409	0.965
6	0.877	0.331	0.974

TABLE II

Variations in A_n

D/d	$n=3$	$n=4$	$n=5$	$n=6$
2	1.081	1.058	1.046	1.032
3	1.076	1.058	1.046	1.032
4	1.074	1.058	1.046	1.032

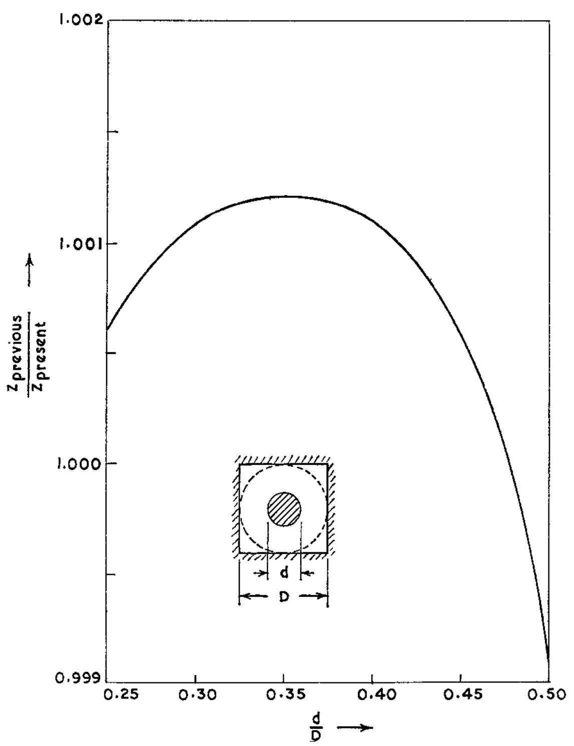


Fig. 2. Comparison of the results of the present method for for $n=4$ with those of an earlier method.

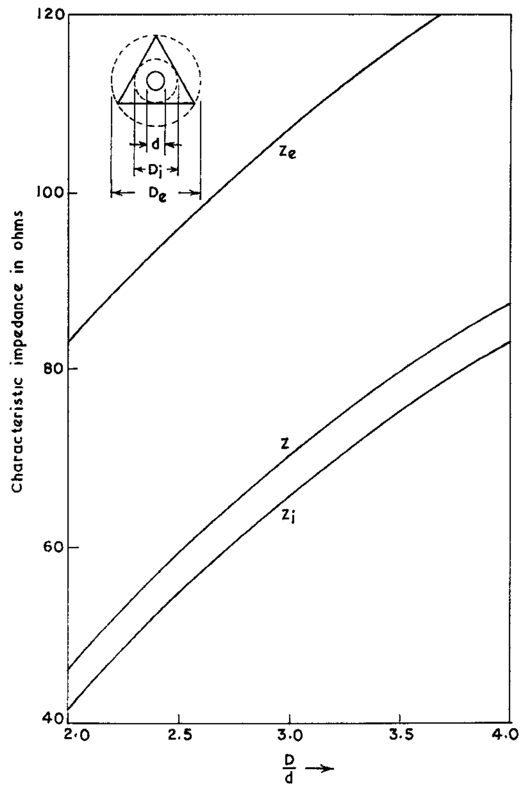


Fig. 3. Characteristic impedance for $n=3$.

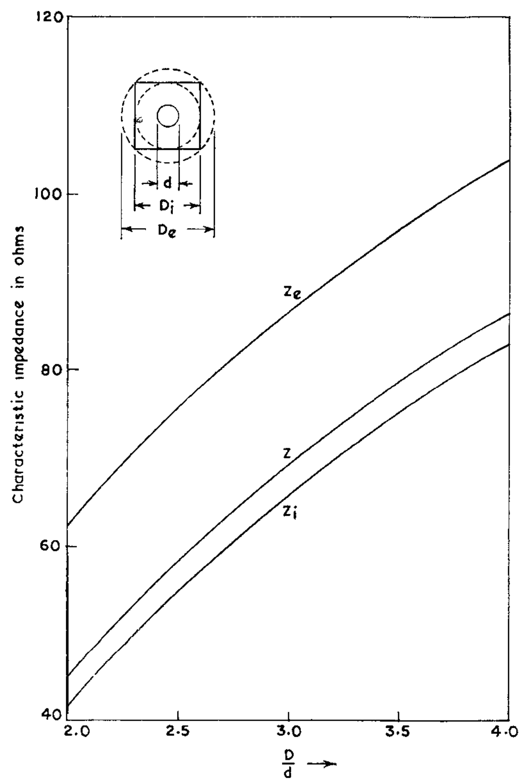


Fig. 4. Characteristic impedance for $n=4$.

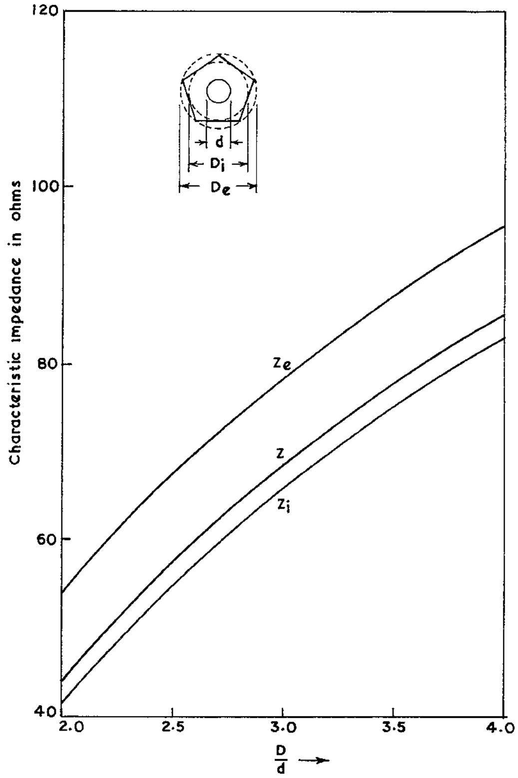


Fig. 5. Characteristic impedance for $n=5$.

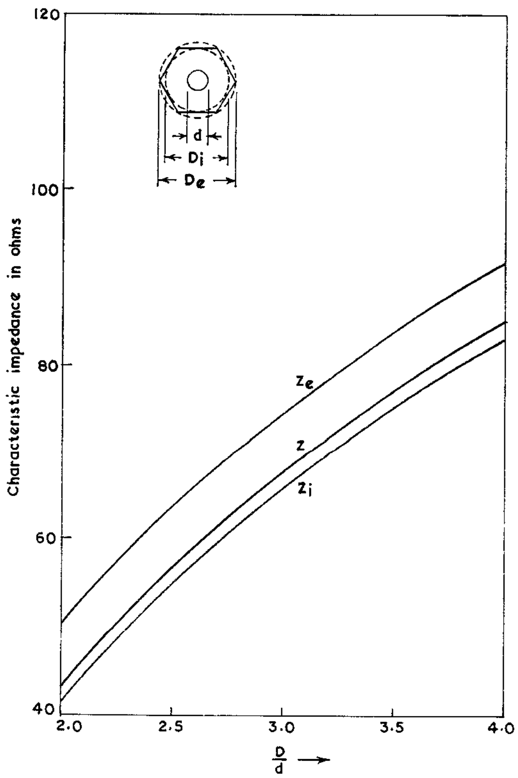


Fig. 6. Characteristic impedance for $n=6$.

B. TEM Wave Equation

The previously mentioned derivations, based on the deformation procedure in conjunction with the analysis for the TEM wave of the elliptic-circular structures previously reported [3], can be extended to derive the TEM wave equation of the polygon-circle configuration if certain restrictions are imposed on the original configuration. It can be seen from the properties of the conformal transformation $W=Z^n$ that if the polygon has its corners rounded off by a factor of the diagonal given by $(1-\sqrt[n]{a_0+e_0})$, then the transformed Riemann sheets become very nearly ellipses so that the TEM wave equation of the structure can be determined. It can be shown by modifying the TEM wave equation of the elliptic-circular line derived earlier, that if $\boldsymbol{E}_{rms}$ is the rms complex electric field strength; $\boldsymbol{H}_{rms}$ is the rms complex magnetic field strength; $\boldsymbol{a}$ is a unit vector; $\boldsymbol{v}_{rms}$ is the rms voltage at $Z=0$; z is the distance along the z axis; u_1 is the value of u at the outer boundary; u_2 is the value at the inner boundary; and η_0 is the intrinsic impedance; then

$$\begin{Bmatrix} \boldsymbol{E}_{rms} \\ \boldsymbol{H}_{rms} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{a}_u \\ \dfrac{\boldsymbol{a}_v}{\eta_0} \end{Bmatrix} \frac{4a_r \boldsymbol{v}_{rms} e^{-j\beta z}}{(-G)\sqrt{D}}. \tag{19}$$

In (19), $G=Z_0/60$ is given by (15). If $(\eta,\ \theta,\ z)$ refers to the polar coordinate of the original polygon structure, $(n\ \Theta)$ is the remainder when $(n\theta)$ is divided by π radians, and

$$\Lambda = \frac{\eta^n + R'r_0 - r_0}{R'};\ w = (a_r^2 + r_0^2)^{\frac{1}{2}}$$

$$a_r = \frac{1}{2e_0}\left[e_0^4 - 2e_0^2(a_0^2 + r_0^2) + (a_0^2 - r_0^2)^2\right]^{\frac{1}{2}}$$

$$A = \left[\frac{b_0^2 e_0 \cos n\Theta + a_0 b_0\{(a_0^2 - e_0^2)\sin^2 n\Theta + b_0^2\cos^2 n\Theta\}^{\frac{1}{2}}}{b_0^2\cos^2 n\Theta + a_0^2 \sin^2 n\Theta}\right]$$

$$B = e_0 \cos n\Theta + (a_0^2 - e_0^2 \sin^2 n\Theta)^{\frac{1}{2}};\ R' = (A - r_0)/(B - r_0)$$

$$u = \sinh^{-1}\left[\frac{2a_r(\Lambda \sin n\Theta + w_1)}{\sqrt{D}}\right]$$

$$v = \sin^{-1}\left[\frac{2a_r(\Lambda \sin n\Theta)}{\sqrt{D}}\right], \tag{20}$$

then it follows from the foregoing that the expression for D is given by

$$D = [\Lambda^2 + w_1^2 + 2w_1\Lambda \cos n\Theta]^2 + a_r^4 - 2a_r^2[\Lambda^2 \cos 2n\Theta + w_1^2 + 2w_1\Lambda \cos \Theta]. \tag{21}$$

IV. Synthesis of Transmission Lines

In actual design problems, it is more useful if the invariant transformation can be obtained when the coefficients are required to satisfy certain prescribed conditions. For example, if a transmission line structure with a given characteristic impedance is to be designed with a prescribed surface area or enclosed volume, some cross section satisfying the given characteristic impedance may be initially designed; and by an invariant transformation subject to the above constraints, it is possible to find the optimum cross section satisfying all at the same time. The problem is one of synthesis because the shape and dimensions of the structure are not known initially, whereas the electrical and mechanical characteristics or the unique parameters are given a priori. In what follows, four typical constraints that may be encountered during synthesis will be considered.

Case 1. Nonintegral Constraint

To find the form of curve C determined by

$$Y_C = \sum_{P=0}^{n} C_P x^P; \qquad (x_1 \le x \le x_2) \tag{22}$$

that has the same characteristic impedance with curve A as the structure characterized by the two plates having cross-sectional equations,

$$\begin{aligned} \text{curve } A\text{:} \quad Y_A &= \sum_{P=0}^{n} A_P x^P; \qquad x_1 \le x \le x_2 \\ \text{curve } B\text{:} \quad Y_B &= \sum_{P=0}^{n} B_P x^P; \qquad x_1 \le x \le x_2, \end{aligned} \tag{23}$$

but satisfying the constraints

$$F_k(C_1, C_2, \cdots, C_n) = 0; \qquad k = 1, 2, \cdots, l. \tag{24}$$

Least-weighted-square deformation will be realized for

$$\frac{\partial}{\partial A_P}\left[\int_{x_1}^{x_2}\left(\sum_{P=0}^{n} B_P x^P - \sum_{P=0}^{n} C_P x^P\right)^2 \frac{dx}{[d(x)]^{2s}}\right] + \sum_{k=1}^{l} \lambda_k \frac{\partial F_k}{\partial A_P} = 0 \tag{25}$$

(λ_k being the undetermined multipliers). Differentiation under integral sign provides $(n+1)$ equations that together with the l equations of the constraint provide $(n+l+1)$ equations in $(n+l+1)$ unknowns, that, in general, will be nonlinear simultaneous equations whose solutions could be obtained to desired accuracy by known methods.

Case 2. Area Constraint

In the above case, if, instead of the general constraints, only the constraint that the area between curves A and C is different from that between A and B by a specified amount is invoked, then the corresponding constraint will be

$$\begin{aligned} \alpha &= \int_{x_1}^{x_2} \sum_{P=0}^{n} (A_P - C_P) x^P \, dx \\ &= \sum_{P=0}^{n} (A_P - C_P) \frac{(x_2 - x_1)^{P+1}}{P+1}. \end{aligned} \tag{26}$$

The least-weighted square is realized for

$$\frac{\partial}{\partial C_P}\left[\int_{x_1}^{x_2}\left\{\sum_{P=0}^{n} (B_P - C_P) x^P\right\}^2 \frac{dx}{[d(x)]^{2s}}\right] + \lambda \frac{\partial}{\partial C_P}\left[\sum_{P=0}^{n} (A_P - C_P)\frac{(x_2 - x_1)^{P+1}}{P+1} - \alpha\right] = 0, \tag{27}$$

which, for any assumed value of the Lagrange's multiplier λ, will be eliminated to yield $(n+1)$ equations in $(n+1)$ unknowns from which the values of C_P can be determined.

Case 3. Integral Constraint

If the deformation is subject to general integral constraints and not algebraic as in (24), (25) can be realized in a similar manner. Here, F_k being an integral may require the differentiation to be carried out under the integral sign.

Case 4. Degree Reduction Constraint

This is an important constraint because it facilitates a design automation in view of the simplicity in synthesizing a class of transmission lines on the digital computer when this constraint is imposed. In the following, a description of this class of structures together with a description of the algorithm is given.

The problem is essentially one of deforming a given structure for which the characteristic impedance is known into another for which the characteristic impedance remains unaltered and that satisfies certain specified electrical and mechanical properties. It is well known that in Cartesian coordinates a polynomial or a power series in x for y can represent a boundary described by a single-valued function, whereas in polar coordinates such a description is possible if r is a polynomial or a power series in θ in the range $0 \le \theta \le 2\pi$. It is also known that many practical transmission lines fall within this description. Even if the given function is involved, the well-known least-square fit can be made to obtain the corresponding polynomial. Therefore, the problem is reduced to one of altering the degree and the value of the coefficients of the given polynomial that satisfies certain electrical and mechanical specifications and for which characteristic impedance remains an invariant. Since the algorithm is essentially the same for both Cartesian and polar coordinates, in the following, its description is confined to the latter.

Let the initial boundaries be defined by

$$R(\theta) = \sum_{r=0}^{n} A_r \theta^r \quad \text{(outer conductor)} \tag{28}$$

$$r(\theta) = \sum_{r=0}^{n} a_r \theta^r \quad \text{(inner conductor)} \tag{29}$$

where $0 \le \theta \le 2\pi$, and for any θ, $R(\theta) > r(\theta)$. It is required to deform the structure, such that

1) the outer boundary can be described by

$$R_f(\theta) = \sum_{r=0}^{m+1} A_{fr} \theta^r, \tag{30}$$

where, in general, $A_{fr} \neq A_r$ and $m < n$;

2) the characteristic impedance remains an invariant;
3) the final structure satisfies some specified mechanical and electrical properties described by

$$f_i(A'_1, \cdots, A'_{m+1}) = 0; \qquad i = 1, \cdots, m. \tag{31}$$

This condition involves m equations in $(m+1)$ unknowns that can have several sets of solutions. However, they can give a unique set of solutions if condition 2) is also considered.

The method consists of three parts. The first step requires the evaluation of $A'_1, \cdots, A'_m$ assuring $A'_{m+1}=0$ in (31) and fits the given boundary to a polynomial of degree n greater than m in which $A'_1, \cdots, A'_m$ have this value. Such a curve fitting can be done easily by the method of least squares [1]. The second step reduces the degree of the polynomial $R(\theta)$ from n to $m+1$ by the least-weighted-square invariance-deformation method. The reduction is carried out in $(n-m)K$ discrete steps, where K is the number of reductions made in the coefficient A_r (successively for $r=n, n-1, \cdots, m+2$) in order to make them assume zero values. As an illustration to the procedure, the first of such reductions will be described. The outer boundary, after reduction, will be

$$R' = \sum_{r=0}^{m} A'_r\theta^r + \sum_{r=m+1}^{n-1} A'_r\theta^r + \left(A_n - \frac{A_n}{K}\right)\theta^2. \tag{32}$$

The corresponding normal equations for the invariance deformation will be

$$\|A'_j\|_j^T = \left\| \int_0^{2\pi} \frac{\theta^{i+j-2}\, d\theta}{[d(\theta)]^{2s}} \right\|^{-1} \left\| \int_0^{2\pi} \frac{\left[R(\theta) - \sum_{r=0}^{m} A'_r\theta^r - \left(A_n - \frac{A_n}{K}\right)\theta^n\right]\theta^{i-1}\, d\theta}{[d(\theta)]^{2s}} \right\|_i^T;$$
$$j, i = m+1, \cdots, n-1. \tag{33}$$

A'_j $(j=m+1, \cdots, n-1)$, calculated from (33) are the new values of A'_r $(r=m+1, \cdots, n-1)$, respectively, which should be substituted into (32), for effecting the next reduction. Equation (32) will then be

$$R'' = \sum_{r=0}^{m} A'_r\theta^r + \sum_{r=m+1}^{n-1} A''_r\theta^r + \left(A_n - \frac{A_n}{2K}\right)\theta^2. \tag{34}$$

The procedure is repeated k times until the coefficient of θ^n becomes zero. Subsequently, (29) is rewritten

$$R_1 = \sum_{r=0}^{m} A'_r\theta^r + \sum_{r=m+1}^{n-2} A'_{r2}\theta^r + \left(A_{n-1,2} - \frac{A_{n-1,2}}{K}\right)\theta^n. \tag{35}$$

The procedure is repeated until the degree of the polynomial reduces to $(m+1)$. In the third step, the value of the coefficient of θ^{m+1} so realized is substituted into the first step. The foregoing procedure is iterated until all of $A'_1, \cdots, A'_{m+1}$ attain consistent values. This method gives a fast convergence for deformations that are not very large. An illustration of the reduction method along with its computer schematic is given in the Appendix where it is used in the determination of the universal constant.

Example

The following is a representative example that describes a synthesis problem. The geometry is chosen arbitrarily in order to emphasize the generality of the procedure.

A transmission line has been formed with one conductor having a cross section represented in 3 dimensions by $y=6-2x+x^2$ $(0 \le x \le 3; 0 \le z \le \infty)$, and another conductor, $y_1=0$ $(0 \le x \le 3; 0 \le z \le \infty)$. It is proposed to design another transmission line having the same characteristic impedance as the above derived from a new conductor, $y_2=a_0+a_1x+a_2x^2$ $(0 \le x \le 3;\ 0 \le z \le \infty)$, with the same conductor, $y_1=0$, subject to the constraint, $a_0+8a_1+21.33a_2=10$ (a volume constraint). The annulus is assumed to be a perfect dielectric.

Application of the least-weighted-square invariance-deformation principle yields

$$\sum_{i=1}^{m}\left[\frac{a_0x_i+a_1x_i^2+a_2x_i^3-6x_i^{j-1}+2x_i^j-x_i^{j+1}}{(6-2x_i+x_i^2)^{2s}}\right] + g_j\lambda = 0; \qquad (j = 1, 2, 3)$$

$$g_1 = 1, \quad g_2 = 8, \quad g_3 = 21.33.$$

Here, let $(x_i - x_{i-1})$ be a constant equal to 0.2. Then, in the range $0 \le x \le 3$, $m=15$. The above three linear equations together with the constraint give, on solving them, the coefficients a_0, a_1, and a_2. The new conductor, consequently, has the description

$$y_2 = 5.721 - 2.111x + 0.992x^2 \ (0 \le x \le 3; 0 \le z \le \infty).$$

V. Transmission Lines with more than Two Conductors

The analysis of structures with more than two boundaries can be carried out approximately by a principle of superposition. On being required to deform a structure with arbitrary boundaries on an all-circle configuration (Fig. 7) such that the capacitances between the conductors are preserved, one must develop methods so as to make $C_{jk}=C_{j'k'}$ $(j \ne k, j' \ne k', jk \ne kj, j'k' \ne k'j')$, where the concerned capacitances are in the presence of the other conductors. Further, let C'_{jk} and $C'_{j'k'}$ $(j \ne k, j' \ne k', jk \ne kj, j'k' \ne k'j')$ be the capacitance between the jth and kth or j'th and k'th conductors in the absence of all other conductors. Superposition, where admissible, assures that if $C'_{jk}=C'_{j'k'}$ is true, then $C_{jk}=C'_{j'k'}$ is true. Since it is the latter that is to be estimated, the former

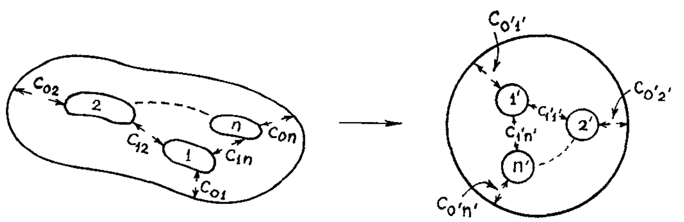


Fig. 7. Transformation of boundaries in lines with more than two conductors.

is only a guide to obtain the all-circle configuration for which both are true. Under such circumstances, superposition is a logical assumption in field theory. Thus, by the least-weighted-square deformation method, the values of C'_{jk} can be determined, which have the same values as $C'_{j'k'}$ in the all-circle structure. If the radii $r_{j'}$ ($j'=0$ to n) and the distance apart of their centers $d_{j'k'}$ ($j' \neq k'$, $j'k' \neq k'j'$) of the conductors of this structure are left undetermined, more equations than unknowns result. As there should be unique solutions, any n of the $d_{j'k'}$ can be assigned positive values, and r_0 can be arbitrarily specified. Thus, from the $(n+1)_{c_2}$ equations [6]

$$r_{j'}^2 + r_{k'}^2 \pm d_{j'k'}^2 = 2r_{j'}r_{k'} \cosh\left(\frac{2\pi\varepsilon}{C'_{j'k'}}\right); \quad j' \neq k', j'k' \neq k'j' \quad j' = 0 \text{ to } n, k' = 0 \text{ to } n, \tag{36}$$

the remaining values can be solved. [In (36) the sign attached to $d_{j'k'}^2$ is positive or negative depending on whether the conductors j' and k' do not or do enclose each other.] The dimensions of the circles as well as their orientations thus being fixed, the capacitances $C_{j'k'}$ can be calculated by known methods that in turn are equal to C_{jk} for the original structure.

VI. Conclusion

A new principle called "least-weighted-square deformation" for deforming transmission line cross sections in such a way as to keep some unique parameters of the structure like the characteristic impedance invariant under such deformations was described. The principle was applied to derive the characteristic impedance, and the TEM wave equations of transmission lines with a regular polygonal conductor and the results were compared with those derived from an earlier method for the case of a square. The synthesis procedure by which the size and shape of the cross section of a transmission line can be derived for given values of electrical and mechanical parameters cannot be obtained easily by earlier methods, for example, those developed by Black and Higgins [7], whereas they are realized by the methods developed in this paper. The method has been extended to boundary value problems in general for calculating field theoretic parameters and to problems requiring arbitrarily high accuracies by introducing a set of universal constants for each field equation. The details concerning this, however, are discussed in a separate communication. It appears that further developments on the synthesis problem on the lines described may give rise to methods of synthesis for several other field structures of microwave engineering.

Appendix

A. Justification for the Theorem

The theorem given in Section II implies the provability of the following three lemmas.

Lemma 1: The least Newtonian path (LNP) length set is unique for the set of points on the perturbed boundary of a uniquely described doubly connected, mutually compatible structure having a unique characteristic impedance.

Lemma 2: It is possible to find a set of weightages for the boundary points on the perturbed boundary that can transform to another, preserving the invariancy of the characteristic impedance.

Lemma 3: The form of weightage is the same function of the LNP length for any invariant deformation.

The justification for the above statements are given in the following analysis.

1) The structure considered has fixed boundaries, is doubly connected, and is mutually compatible. The last requirement implies, by definition, that the endpoints of the LNP trace both the boundaries continuously and entirely. Also, corresponding to every point on the perturbed boundary there is a unique LNP. Conversely, given the set of lengths of the LNP, there corresponds a unique configuration of the structure. Since, for uniquely described boundaries, there is a unique characteristic impedance (for TEM), the first statement results.

2) Since two structures with two inclined plates of finite breadth and infinite length having the angle subtended between the plates slightly different from one another, can, in general, be adjusted to yield the same capacitance or characteristic impedance, a necessary requisite would be that the two structures with one plate common, when superposed, would have the inclined plates intersecting somewhere along the breadth of the lines. Thus, for a transformation preserving the invariancy, the deformation, in general, will not be the best fit curve fitted by a least-square method. This is true for other structures, also. If the least-square principle is used as a medium of deformation, in order that the deformation will not be the best fit, different relative weights at every point of the perturbed boundary are, in general, necessary to obtain the transformation.

Further, if a weightage is assigned to every point on the perturbed boundary, then any change of weight at any point on the perturbed boundary would, in general, affect the deformation of every other point. This is true because the least squares are computed only after reckoning the entire boundary.

If deformation of the unperturbed boundary in part or whole takes place, then, for an invariant deformation, the weightage, in general, has to change on the perturbed boundary and conversely. This follows from the fact that the deformation of the unperturbed boundary changes the weightage at least at one point on the perturbed boundary, consequently affecting the least-weighted-square deformation everywhere on the perturbed boundary, owing to the previously mentioned property.

We thus obtain the second statement.

3) The last statement can be established as follows.

a) It is possible to obtain the weightages as some function of the LNP: It is known that the constraints like characteristic impedance or capacitance can be obtained as the function of the structural dimensions. From Lemma 1, it follows that an entire set of lengths of LNP can specify a structure uniquely. The constant parameters may thus be considered as functions of the LNP set. The invariant transformation thus depends on the nature of weightages. Since the LNP requires that the constant parameter be kept an invariant, and since by Lemma 2 such a transformation exists, it is possible to obtain the weightages as some function of LNP.

b) The greater the LNP, the less the weightage and vice versa: This is a consequence of the retarding potential. It is known that the potential from a charged region decreases as the distance from the region increases. Since LNP is related to distance and the weightage to potential, the above statement is consequential.

c) The superficial magnification leaves the characteristic impedance of the structure an invariant: It is well known that it is true for the capacitance of structures that if all dimensions are multiplied by a dimensionless constant, the capacitance is unaltered. Since the characteristic impedance is inversely proportional to the capacitance, the above statement follows.

d) The least-square fitting multiplied by the same constant weightage for all points results in the same deformation as without the multiplication: This is because, giving equal weight, however large, to each point is the same as giving no weight at all.

e) Superficial magnification leaves the weightage at every point on the perturbed boundary linearly increased by the same proportionality constant: Consequent to step 3 c), the invariant transformation of the modified set of boundaries of the superficially magnified structure by the modified weightages would be a magnification by a same amount of the transformed original structure by the original set of weightage.

f) The function whose weightages are preserved under an invariant transformation has the general form $(1/\text{LNP})^s$, where s is a constant: From 3 a), weightage at point i on the perturbed boundary w_i is given by

$$w_i = [F(d_i)]^{-1},$$

where d_i is the length of LNP at point i.

The simplest functions satisfying 3c) and 3e) are

$$w_i = (kd_i)^{-s} \quad \text{and} \quad (kd_i)^s.$$

The latter alternative is not tenable because of 3b). Hence, from considerations resulting from 3d),

$$w_i = d_i^{-s} = \left[\frac{1}{\text{LNP}}\right]^s.$$

B. Evaluation of s

In the foregoing part of this Appendix, a dimensionless constant s has been introduced. It is the purpose of this section to evaluate this constant numerically for structures governed by the TEM wave, such as transmission lines, when the field theoretic parameter under consideration is the characteristic impedance or the capacitance.

The constant s is evaluated by noting that since the analysis holds good for double-connected power-series-describable mutually compatible structures in general, certain well-known configurations, which have been analyzed previously, can also fit into the category. Thus, two structures, differing by a small amount in one of the boundaries, may be chosen, whose analysis is available in published literature. In the Cartesian coordinates inclined planes and parallel planes may be considered, while in the polar coordinates eccentric and concentric cylinders may be chosen. In the following, the method is illustrated in terms of the latter. A transmission line with a cylindrical inner conductor of radius R_i and a cylindrical outer conductor of radius R_0, whose axes are separated by a distance D, is considered. It is assumed that $R_i = D = 1$ and $R_0 = 15$. The outer conductor is described as a power series in θ by employing the least-square fit. For example, to the third degree, it is given by

$$r = 16.08 - 1.059\theta + 0.1197\theta^2 + 0.0082\theta^3; \quad (0 < \theta < 2\pi).$$

It is assumed that the origin is situated at the center of the inner conductor. The degree-reduction method is employed for reducing the above polynomial to the final form

$$r = c,$$

representing the coaxial transmission lines. The flow schematic shown in Fig. 8 has been included to illustrate the manner in which a degree reduction governed by the invariance-deformation principle can be computerized. The final result sought from the computer being s, it is varied from two downwards and the value at which F_1 and F_2 given by

$$F_1 = \cosh^{-1}\left(\frac{R_i^2 + R_0^2 - D^2}{2R_iR_0}\right); \quad F_2 = \ln\left(\frac{C}{R_i}\right)$$

are equal to a specified tolerance, is taken as the required value. A computation performed on the above structure gave a value of 1.251 for s. It should be mentioned that the

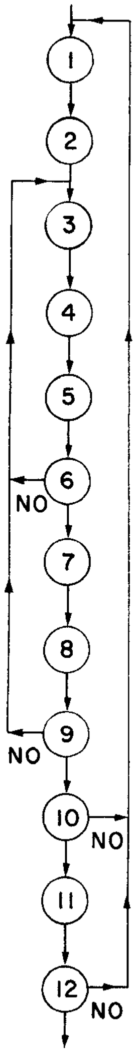


Fig. 8. Flow schematic for the degree-reduction method. Initial values: α, ε, a, R_1, R_2, s, A_j for $j=1$ to N; 1) $j=N$: $A_N''=A_N$; 2) $s \Leftrightarrow s-\alpha$; 3) $A_j \Leftrightarrow A_j - A_j''/10$; 4) A_j' from equation similar to (33); 5) $A_k=A_k'$ for $k=1, 2, \cdots, j-1$; 6) Decision: $A_j=0$?; 7) $j \Leftrightarrow j-1$; 8) $A_j''=A_j$; 9) Decision; $j<2$?; 10) Decision: (F_2-F_1) changes sign?; 11) $\alpha \Leftrightarrow \alpha/10$; 12) Decision: $\alpha<\varepsilon$?

representation of the weightage by only one universal constant is an approximation. In general, the weightage will be

$$w_i = \left(\sum_j \frac{1}{d_{ij}^{2s_0}}\right) + s_1\left(\sum_j \frac{1}{d_{ij}^{s_0}}\right)^2 + s_2\left(\sum_j \frac{1}{d_{ij}^{2s_0/3}}\right)^3 + \cdots,$$

where d_{ij} is the set of Newtonian path lengths and s_0, s_1, s_2, $\cdots$, are a set of universal constants.

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